

# CONSTANT MEAN CURVATURE, FLUX CONSERVATION, AND SYMMETRY

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ABSTRACT. As first observed in [KKS], constant mean curvature implies a homological conservation law for hypersurfaces in ambient spaces with Killing fields. In Theorem 3.5 here, we generalize that law by relaxing the topological restrictions assumed in [KKS], and by allowing a weighted mean curvature functional. We also prove a partial converse (Theorem 4.1). Roughly, it states that when flux is conserved along enough Killing fields, a hypersurface splits into two regions: one with constant (weighted) mean curvature, and one fixed by the given Killing fields. We demonstrate the use of our theory by using it to derive a first integral for twizzlers, i.e, helicoidal surfaces of constant mean curvature in  $\mathbb{R}^3$ .

## 1. INTRODUCTION

Constant mean curvature (“CMC”) imposes a homological flux conservation law on hypersurfaces of ambient spaces with non-trivial Killing fields. This was first observed and exploited by Korevaar, Kusner, & Solomon in their 1989 paper on the structure of embedded CMC surfaces in  $\mathbb{R}^3$  [KKS]. In Theorem 3.5 here, we generalize that law by relaxing the topological restrictions assumed by [KKS], and by allowing a weighted version of the mean curvature functional. We further extend the theory in Theorem 4.1, which gives a partial converse to our conservation law. Roughly, it states that when the appropriate flux is conserved along enough Killing fields, the hypersurface splits into two regions (though either may be empty): a region with constant (weighted) mean curvature, and a region fixed by the given Killing fields.

We apply our results by using them in Section 4.6 to quickly derive the seemingly ad hoc *first integral* that Perdomo [P1], DoCarmo & Dajczer [DD], and others have used to analyze the moduli space of CMC

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*Date:* Begun August, 2011. Latest draft February 14, 2013.

surfaces with helicoidal symmetry, also known as *twizzlers*. Twizzlers have been studied by Wunderlich [W] and Halldorsson [H] as well, and they motivated the results we develop here. In general, constancy of weighted mean curvature is characterized by a non-linear second-order PDE, and its Nöetherian reduction to a first-order condition makes it easier to analyze.

When a CMC hypersurface  $\Sigma$  in a manifold  $N$  is preserved by the action of a continuous isometry group  $\mathcal{G}$ , one can project into the orbit space  $N/\mathcal{G}$ . In this case, the *projected* hypersurface  $\Sigma/\mathcal{G}$  will be stationary for the *weighted* functional introduced in §3.3. We include the weighted functional and the resulting weighted mean-curvature with an eye toward this fact. We suspect that virtually all we do here could be developed in a more general, stratified setting subsuming both riemannian manifolds and their quotients under smooth group actions, but we have avoided the technical issues arising there by staying within the set of smooth ambient manifolds.

In practice, however, one often works in the orbit space, and our case study in §4.6 could easily have been re-interpreted in that context. The approach we demonstrate there can also be adapted to spherical and hyperbolic space forms. The first author's report [E] sketches out one way to do that, but we say a few words about the orbit-space approach to those examples in our final Remark 4.7.2.

## 2. PRELIMINARIES

Let  $N$  denote an  $n$ -dimensional oriented riemannian manifold, and consider a smooth, connected, oriented, properly immersed hypersurface  $f : \Sigma^{n-1} \rightarrow N$ . We often abuse notation by writing  $\Sigma$  when we mean  $f(\Sigma)$  or even  $f : \Sigma \rightarrow N$ , leaving context to clarify the intended meaning.

The *mean curvature function*  $h : \Sigma \rightarrow \mathbb{R}$  is the trace of the shape operator associated with the unit normal  $\nu$  that completes the orientation of  $\Sigma$  to that of  $N$ . Notationally,

$$(2.1) \quad h = \operatorname{div}_{\Sigma}(\nu)$$

Here  $\operatorname{div}_{\Sigma}(Y)$  denotes the *intrinsic divergence* of a vectorfield  $Y$  along  $\Sigma$ , that is, the trace of the endomorphism  $T\Sigma \rightarrow T\Sigma$  gotten at each

$p \in \Sigma$  by projecting the covariant derivative  $\nabla Y$  onto  $T_p \Sigma$ . One may compute  $\operatorname{div}_\Sigma$  locally using any orthonormal basis  $\{e_i\}$  for  $T_p \Sigma$  via

$$\operatorname{div}_\Sigma(Y) := \sum_{i=1}^{n-1} \nabla_{e_i} Y \cdot e_i .$$

**2.1. Chains and  $k$ -area.** The homology of the sequence

$$N \longrightarrow (N, \Sigma) \longrightarrow \Sigma$$

plays a key role below, and makes it inconvenient to work completely in terms of smooth submanifolds. We therefore turn to the following class of piecewise smooth objects:

**Definition 2.1.1.** A **smooth  $r$ -chain** (or simply **chain**) in a smooth manifold  $M$  is a finite union of smoothly immersed oriented  $r$ -dimensional simplices. We regard a chain  $X$  as a formal homological sum:

$$(2.2) \quad X = \sum_{i=1}^m m_i f_i$$

Here each  $f_i : \Delta \rightarrow M$ , immerses the standard closed, oriented  $r$ -simplex  $\Delta$  (along with its boundary) smoothly into  $M$ , and the  $m_i$ 's are integers.

It is trivial to integrate an  $r$ -form  $\phi$  on  $M$  over such a chain: we simply define

$$\int_X \phi := \sum_{i=1}^m m_i \int_\Delta f_i^* \phi$$

where  $f_i^*$  denotes the usual pullback.

Given a riemannian metric on  $M$ , one can also integrate functions over chains, and most importantly for our purposes, compute weighted volumes.

**Definition 2.1.2.** Let  $\mu : M \rightarrow \mathbb{R}$  be any continuous function. Define the  **$\mu$ -weighted  $r$ -volume**  $|X|_\mu$  of the  $r$ -chain  $X$  in (2.2) as

$$|X|_\mu := \sup \left\{ \int_X e^\mu \phi : \phi \text{ is an } r\text{-form on } M \text{ with } \|\phi\|_\infty \leq 1 \right\}$$

For a single embedded simplex, the usual riemannian volume integral gives a simpler definition. To allow coincident, oppositely oriented simplices to cancel, however, we need the subtler definition above<sup>1</sup>.

Write  $\partial X$  for the homological **boundary** of a chain  $X$ . We write  $\mathcal{Z}_i(M)$  and  $\mathcal{B}_i(M)$  respectively, for the spaces of  $i$ -dimensional **cycles** and **boundaries** (kernel and image of  $\partial$ ) in  $M$ . Likewise  $\mathcal{Z}_i(M, A)$  and  $\mathcal{B}_i(M, A)$  are spaces of cycles and boundaries modulo a subset  $A \subset M$ .

Finally, note that because Stokes' Theorem holds for immersed  $r$ -simplices, it holds for  $r$ -chains as well.

**2.2. Symmetry.** Our work here is vacuous unless the ambient space  $N$  has non-trivial Killing fields.

Write  $\mathcal{I}$  and  $L(\mathcal{I})$  respectively for the isometry group of  $N$  and its Lie algebra. Identify  $L(\mathcal{I})$  with the linear space of Killing fields on  $N$  in the usual way, associating each  $Y \in L(\mathcal{I})$  with the Killing field (also called  $Y$ ) we get by differentiating the flow that sends  $p \in N$  along the path  $t \mapsto \exp(tY)p$ . We write  $Y_p$  for the value of  $Y$  at  $p$ .

One often studies CMC hypersurfaces (like surfaces of revolution and twizzlers in  $\mathbb{R}^3$ ) in relation to the action of a closed, connected subgroup  $\mathcal{L} \subset \mathcal{I}$ . Though it complicates our exposition to some extent, the presence of such a subgroup  $\mathcal{L}$ —like that of the density function  $e^\mu$ —allows an interesting increase in generality. Our results generalize those of [KKS] even when  $\mu \equiv 0$  and  $\mathcal{L}$  is the full isometry group of  $N$ , however, and readers may want to keep that in mind.

When we reach Theorem 4.1 (a converse to our Conservation Law) we encounter a situation where the Killing fields associated with  $\mathcal{L}$  remain everywhere tangent to an open subset  $S$  of our hypersurface  $\Sigma \subset N$ . The following Lemma (and its Corollary) then lets us deduce  $\mathcal{L}$ -invariance of  $S$ .

**Lemma 2.3.** *Suppose  $S \subset N$  is a hypersurface, and that for all  $Y \in L(\mathcal{L})$ , we have  $Y_p \in T_p S$  for every  $p \in S$ . Then each  $p \in S$  has a compact neighborhood  $\mathcal{O}_p \subset S$ , associated with an identity neighborhood  $\mathcal{O}_e \subset \mathcal{L}$ , such that  $gq \in S$  for all  $(g, q) \in \mathcal{O}_e \times \mathcal{O}_p$ .*

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<sup>1</sup>Definition 2.1.2 amounts to a weighted version of the *mass* of  $X$  as a *current*, in the sense of geometric measure theory [GMT, p. 358].

*Proof.* Since  $S$  is a submanifold, there exists an open set  $W \subset N$  that contains  $S$ , but no point of  $\bar{S} \setminus S$  ( $\bar{S}$  = closure of  $S$ ). Let  $\Theta : L(\mathcal{G}) \times S \times \mathbb{R} \rightarrow N$  denote the continuous mapping

$$\Theta(Y, q, t) := \exp(tY)q$$

Then  $\Theta^{-1}(W)$  is an open neighborhood of  $L(\mathcal{L}) \times S \times \{0\}$ .

Also, for each  $q \in S$  and  $Y \in L(\mathcal{L})$ ,  $\Theta(Y, q, t)$  parametrizes the integral curve of  $Y$  in  $N$  with initial velocity  $Y_q$ . But  $Y$  is everywhere tangent along  $S$ , and first-order ODE's have unique solutions, so this integral curve must remain in  $S$  for all  $(Y, q, t) \in W$ . In short,  $\Theta^{-1}(S) = \Theta^{-1}(W)$ .

It follows that, given compact neighborhoods  $\mathcal{O}_p$  and  $\mathcal{O}'$  of  $p \in S$  and  $0 \in L(\mathcal{L})$  respectively, there exists a fixed  $\varepsilon > 0$  such that

$$\mathcal{O}' \times \mathcal{O}_p \times (-\varepsilon, \varepsilon) \subset \Theta^{-1}(S)$$

The Lemma then holds with this choice of  $\mathcal{O}_p$ , and  $\mathcal{O}_e := \exp(\varepsilon \mathcal{O}')$ .  $\square$

**Corollary 2.4.** *If  $S \subset N$  is a hypersurface, with  $Y_p \in T_p S$  for each  $p \in S$  and all  $Y \in L(\mathcal{L})$ , then  $Q \cap S$  is open in  $Q$  for each orbit  $Q$  of the  $\mathcal{L}$ -action on  $N$ .*

*Proof.* If  $p \in Q \cap S$ , the Lemma yields  $gp \in S$  for all  $g \in \mathcal{O}_e$ , and the set of all such  $gp$  clearly cover a neighborhood of  $p$  in  $Q$ .  $\square$

**2.5. Flux.** Korevaar, Kusner, and Solomon showed in [KKS] that when a hypersurface  $\Sigma \subset N^n$  has constant mean curvature  $h \equiv H$ , and the homology groups  $H_{n-1}(N)$  and  $H_{n-2}(N)$  are both trivial (over  $\mathbb{Z}$ ; all homology groups in this paper have integer coefficients), there exists a *flux homomorphism*

$$\phi : H_{n-2}(\Sigma) \rightarrow L(\mathcal{I})^*$$

(asterisk signifying *dual*) defined by assigning, to any Killing field  $Y$  and any class  $\mathbf{k} \in H_{n-2}(\Sigma)$ , the **flux**  $\phi(\mathbf{k})(Y)$  of  $Y$  across  $\mathbf{k}$ , where

$$(2.3) \quad \phi(\mathbf{k})(Y) := \int_{\Gamma} \eta \cdot Y - H \int_K \nu \cdot Y$$

Here

- $\Gamma$  can be an  $(n-2)$ -cycle representing  $\mathbf{k}$
- $K \subset N$  can be any  $(n-1)$ -chain bounded by  $\Gamma$
- $\eta$  is the orienting unit conormal to  $\Gamma$  in  $\Sigma$ , and
- $\nu$  is the orienting unit normal to  $K$  in  $N$ .

The conservation law in [KKS] assumes two topological triviality assumptions:  $H_{n-1}(N) = H_{n-2}(N) = 0$ . The vanishing of  $H_{n-2}(N)$  ensures  $\Gamma$  will bound *some* chain  $K$ , while that of  $H_{n-1}(N)$  means any competing chain  $K'$  with  $\partial K' = \partial K$  can be written  $K' = K + \partial U$  for some  $n$ -chain  $U$ . Since Killing fields are divergence-free, the Divergence Theorem then ensures that second integral in (2.3) is independent of our choice of  $K$ .

Here, we will extend the [KKS] flux result in several ways.

First, in §3.3, we broaden the mean curvature functional by allowing  $\mu$ -weighted area and volume as in Definition 2.1.2. This is a minor tweak of the standard theory, but it does *not* correspond to a mere conformal change of metric, since  $n$ - and  $(n-1)$ -dimensional volume scale differently under conformal change. Moreover, the  $\mu$ -weighted theory has a geometric application: it relates the geometry of equivariant CMC hypersurfaces in  $N$  with that of hypersurfaces in the orbit space  $N/\mathcal{L}$ . See Remark 3.3.1 below.

Second and more importantly, however, we remove the homological triviality requirements on  $N$  discussed above. Though we follow the same variational strategy used in [KKS], we remove their topological constraints by migrating the flux invariant to a certain *relative* homology group. Specifically, in Definition 3.0.2, we introduce an  $(n-2)$ -dimensional “reference” set  $B \subset N$ , called a *spine*, that generates the same subgroup of  $H_{n-2}(N)$  as  $\Sigma$ . We then realize our invariant in  $H_{n-1}(N, \Sigma \cup B)$  instead of  $H_{n-2}(\Sigma)$ . The new theory reduces to the old one when  $H_{n-1}(N) = H_{n-2}(N) = 0$ , for there we can take  $B = \emptyset$ , whence  $H_{n-1}(N, \Sigma \cup B) = H_{n-1}(N, \Sigma)$ , which is in turn isomorphic to  $H_{n-2}(\Sigma)$  because the relevant piece of the long exact sequence for  $(N, \Sigma)$  becomes short exact when  $H_{n-1}(N) = H_{n-2}(N) = 0$ :

$$H_{n-1}(N) \longrightarrow H_{n-1}(N, \Sigma) \xrightarrow{\partial} H_{n-2}(\Sigma) \longrightarrow H_{n-2}(N) .$$

We work out the resulting generalized conservation law in §3, and then, in §4, develop a partial converse.

Before proceeding to these extensions however, we describe a motivating example. We shall return to it later as an illustration.

**Example 2.5.1.** *Twizzlers* are “helicoidal” CMC surfaces in  $\mathbb{R}^3$  — surfaces invariant under a 1-parameter group of screw motions. Any such surface can be gotten by applying the screw motion to a curve  $\gamma$  in a plane perpendicular to the screw-axis, and the resulting helicoidal surface will then have mean curvature  $h \equiv H$  if and only if  $\gamma$  satisfies an easily-derived second order ODE. As others ([DD], [W], [P1], [H]) have noted, however, that ODE has a useful first integral. We derive that first integral using flux conservation below.

The conservation law as formulated in [KKS] yields nothing for twizzlers, however. Indeed, the generic CMC twizzler is generated by a non-periodic curve  $\gamma$  in the transverse plane, and thus lacks homology. To remedy that, one can mod out by the translational period of the helicoidal motion, realizing the twizzler as an immersion of a *cylinder* in  $N := \mathbb{R}^2 \times \mathbf{S}^1$ . Cylinders *do* have non-trivial loops, but those loops don’t bound in  $N$ , and hence can’t be capped off as required by [KKS].

Our approach bypasses the obstruction, as we shall see in Example 3.1.2 and §4.6.  $\square$

### 3. CONSERVATION

Like [KKS], we derive flux-conservation using a constrained first-variation formula. We make two notable modifications, however.

First, we weight both the areas of hypersurfaces and the volumes of domains by an  $\mathcal{L}$ -invariant *density function*

$$(3.1) \quad e^\mu : N \longrightarrow (0, \infty)$$

Here  $\mu$  can be any smooth function fixed by  $\mathcal{L}$ . The formula in [KKS] corresponds to the case  $\mu \equiv 0$ , as will become clear in §3.3 below.

Secondly, we encode the  $(n-2)$ -dimensional homology of our immersion  $f : \Sigma \rightarrow N$  into a fixed “reference” set  $B \subset N$  that we call a *spine*. Writing  $i$  for the inclusion  $B \subset N$ , let  $f_*$  and  $i_*$  respectively denote the induced homomorphisms

$$H_{n-2}(\Sigma) \xrightarrow{f_*} H_{n-2}(N), \quad H_{n-2}(B) \xrightarrow{i_*} H_{n-2}(N)$$

**Definition 3.0.2** (Spine). We call a subset  $B \subset N$  a **spine** for the pair  $(N, \Sigma)$  when the inclusion  $i : B \rightarrow N$  has these properties:

- a)  $i(B)$  is locally compact, with locally finite  $(n - 2)$ -dimensional Hausdorff measure.
- b)  $H_{n-2}(B) \xrightarrow{i_*} H_{n-2}(N)$  is injective.
- c)  $i_* H_{n-2}(B) = f_* H_{n-2}(\Sigma)$ .

**Lemma 3.1.** *When  $f : \Sigma \rightarrow N$  is a smooth immersion,  $N$  contains a spine  $B$  for  $(N, \Sigma)$ . Any cycle  $\Gamma \in \mathcal{Z}_{n-2}(\Sigma)$  is then homologous to a unique  $\beta \in \mathcal{Z}_{n-2}(B)$ .*

*Proof.* Let  $A$  denote an independent set in  $H_{n-2}(\Sigma)$  that generates  $f_* H_{n-2}(\Sigma)$ . Each class in  $A$  contains a mass-minimizing integral current whose support is a compact rectifiable set with finite  $(n - 2)$ -dimensional Hausdorff measure [GMT, §5.1.6]. The union of supports of these cycles then constitutes a spine  $B$  for  $(N, \Sigma)$ .

Now consider any cycle  $\Gamma \in \mathcal{Z}_{n-2}(\Sigma)$ . Property (c) from Definition 3.0.2 ensures that  $B$  supports a cycle  $\beta$  homologous to  $\Gamma$ . Property (b) ensures that any other cycle  $\beta'$  in  $\mathcal{Z}_{n-2}(B)$  with  $\beta' \sim \Gamma$  in  $N$  also satisfies  $\beta' \sim \beta$  in  $H_{n-2}(B)$ , so that  $\beta' - \beta = \partial X$  for some  $(n - 1)$ -chain  $X$  in  $B$ . But  $B$  has Hausdorff dimension  $n - 2$ , and our chains are sums of immersed simplices, so  $X = 0$ . It follows that  $\beta' = \beta$ , i.e.,  $\beta$  is uniquely determined by  $\Gamma$ .  $\square$

**Definition 3.1.1** (Cap). Suppose  $B$  is a spine for  $(N, \Sigma)$ , and that  $\Gamma \in \mathcal{Z}_{n-2}(\Sigma)$ . Let  $\beta$  be the unique  $(n - 2)$ -cycle in  $B$  that is homologous to  $\Gamma$  in  $N$ , as provided by Lemma 3.1. Then there exists an  $(n - 1)$ -chain  $K$  in  $N$  (by no means unique) such that

$$\partial K = \beta - \Gamma$$

We call any such  $K$  a **cap** for  $\Gamma$ . Note that caps are  $(n - 1)$ -cycles for the pair  $(N, \Sigma \cup B)$ . The corresponding relative homology group  $H_{n-1}(N, \Sigma \cup B)$  will host our flux invariant, as mentioned in §2.5 above

We can illustrate our notions of *spine* and *cap* using twizzlers:

**Example 3.1.2.** As explained in Example 2.5.1, we may regard a twizzler as a cylinder  $\Sigma = \mathbb{R} \times \mathbf{S}^1$  immersed in  $N = \mathbb{C} \times \mathbf{S}^1$  and preserved by the helical  $\mathbf{S}^1$  action

$$(3.2) \quad [e^{i\theta}] (z, e^{it}) = (e^{i\theta} z, e^{i(t+\theta)})$$



The length of the  $\mathbf{S}^1$  factor is geometrically significant, but the standard metric with length  $2\pi$  will serve our purposes here.

We call the orbits of the  $\mathbf{S}^1$  action *helices*. By construction, both  $N$  and the twizzler  $f(\Sigma)$  are foliated by such helices, any one of which generates  $f_*H_1(\Sigma) = H_1(N)$ . It follows that any helix in  $N$  qualifies as a spine for  $(N, \Sigma)$ . We take the shortest one, namely  $\mathbf{0} \times \mathbf{S}^1 \subset N$ , as our spine  $B$ .

Suppose a twizzler is generated by a particular curve  $\gamma : \mathbb{R} \rightarrow \mathbb{C}$ , and that we immerse  $\Sigma$  in  $N$  accordingly, via

$$f(t, e^{i\theta}) = (e^{i\theta}\gamma(t), e^{i\theta})$$

For each fixed  $t \in \mathbb{R}$ , the helix  $\Gamma_t := f(t, \mathbf{S}^1)$  forms a non-trivial cycle in  $H_1(\Sigma)$ . Any oriented surface that realizes the homology between  $\Gamma_t$  and the compatibly oriented cycle  $\beta \in \mathcal{Z}_2(B)$  is then a *cap* for  $\Gamma_t$ .

For instance, the line segment (indeed, any arc) joining  $\mathbf{0}$  to  $\gamma(t)$  in  $\mathbb{C}$  will, under the  $\mathbf{S}^1$  action 3.2, sweep out a cap. Such caps are preserved by the  $\mathbf{S}^1$  action, a useful property that many other caps lack.  $\square$

To proceed toward our first-variation formula, fix a spine  $B$  for  $(N, \Sigma)$ , and suppose we have two smooth, *homologous* cycles  $\Gamma, \Gamma' \in \mathcal{Z}_{n-2}(\Sigma)$ . Lemma 3.1 then asserts the existence of a unique  $\beta \in \mathcal{Z}_{n-2}(B)$  that satisfies

$$\beta \sim \Gamma \sim \Gamma' \quad \text{in } H_{n-2}(N)$$

and we can therefore find caps  $K, K'$  in  $N$  for  $\Gamma$  and  $\Gamma'$  respectively, with

$$\partial K = \beta - \Gamma \quad \text{and} \quad \partial K' = \beta - \Gamma'.$$

Both  $K$  and  $K'$  represent classes in  $H_{n-1}(N, \Sigma \cup B)$ .

**Observation 3.2.** *When  $K \sim K'$  in  $H_{n-1}(N, \Sigma \cup B)$ , there exists an  $(n-1)$ -chain  $S$  in  $\Sigma$  so that  $S + K - K' = \partial U$  for some  $n$ -chain  $U$  in  $N$ .*

*Proof.* Our hypothesis means that  $K - K'$  bounds in  $N$  modulo  $(n-1)$ -chains in  $\Sigma \cup B$ . But  $B$  has dimension  $n-2$ , so its  $(n-1)$ -chains vanish homologically. It follows that  $K - K' = \partial U + S$ , for some  $n$ -chain  $U$  in  $N$ , and some  $(n-1)$ -chain  $S$  in  $\Sigma$ .  $\square$

**3.3. First variation.** In the situation just described by Observation 3.2, and in the presence of a log-density function  $\mu$ , we may consider the  $n$ - and  $(n-1)$ -dimensional  $\mu$ -weighted volumes  $|U|_\mu$  and  $|S|_\mu$  of the chains  $U$  and  $S$  respectively (Definition 2.1.2). We want to deform these chains along the flow of an arbitrary smooth vectorfield  $Y$ , fix a scalar  $H$ , and consider the initial derivative of  $|S|_\mu - H |U|_\mu$  with respect to this flow, written

$$(3.3) \quad \delta_Y \left( |S|_\mu - H |U|_\mu \right)$$

We call this the **( $\mu$ -weighted) volume-constrained first-variation** of  $S$ , and obtain our conservation law by assuming constant mean curvature  $h \equiv H$  for  $\Sigma \supset S$ , and then evaluating (3.3) on the Killing vectorfields of  $N$ . To manage the calculation, we first analyze  $\delta_Y |S|_\mu$  and  $\delta_Y |U|_\mu$  separately.

Starting with the latter, recall that when  $\mu \equiv 0$ , a standard calculation finds  $\delta_Y |U|_\mu$  equal to the integral of  $\text{div}_N(Y)$  over  $U$ . A routine modification of that calculation shows that for arbitrary  $\mu$ , we have

$$\delta_Y |U|_\mu = \int_U \text{div}_N(e^\mu Y) = \int_{\partial U} e^\mu Y \cdot \nu,$$

where  $\nu$  denotes the orienting unit normal along  $\partial U$ . Since  $\partial U = S + K - K'$ , we can rewrite this as

$$(3.4) \quad \delta_Y |U|_\mu = \int_S e^\mu Y \cdot \nu + \int_{K-K'} e^\mu Y \cdot \nu$$

A similar modification of the  $\mu \equiv 0$  case as analyzed in [SL, pp. 46–51], computes the  $\mu$ -weighted first-variation of  $|S|_\mu$  along  $Y$ :

$$(3.5) \quad \delta_Y |S|_\mu = \int_S e^\mu d\mu(\nu) \nu \cdot Y + \text{div}_\Sigma(e^\mu Y^\top) + \text{div}_\Sigma(e^\mu Y^\perp)$$

Here  $Y^\top$  and  $Y^\perp$  signify the tangential and normal components, respectively, of  $Y$  along  $S$ .

For vectorfields *tangent* to  $\Sigma$ , the Divergence Theorem applies in its usual form: Given an  $(n-1)$ -chain  $S$  in  $\Sigma$  with oriented unit conormal  $\eta$  along its boundary, we have

$$\int_S \text{div}_\Sigma(X) = \int_{\partial S} X \cdot \eta \quad (X \text{ tangent to } \Sigma)$$

The divergence of a vectorfield  $Z$  *normal* to  $\Sigma$ , on the other hand, brings the mean curvature function  $h$  of  $\Sigma$  into play, thanks to (2.1).

For a purely normal vectorfield  $Z$ , we have  $Z = (Z \cdot \nu)\nu$ , whence the Leibniz rule yields

$$\int_S \operatorname{div}_\Sigma(Z) = \int_S (Z \cdot \nu) h \quad (Z \text{ normal to } \Sigma)$$

Accordingly, let us define the  $\mu$ -**mean curvature**  $h_\mu$  along  $\Sigma$  as

$$h_\mu := h + d\mu(\nu) .$$

With this notation, the facts above reduce (3.5) to

$$(3.6) \quad \delta_Y |S|_\mu = \int_{\partial S} e^\mu Y \cdot \eta + \int_S e^\mu h_\mu Y \cdot \nu$$

Finally, using (3.4), (3.6), and writing  $\partial S = \Gamma - \Gamma'$ , we can put our volume-constrained first-variation formula (3.3) into the form we need:

$$(3.7) \quad \begin{aligned} & \delta_Y (|S|_\mu - H |U|_\mu) \\ &= \int_{\Gamma - \Gamma'} e^\mu \eta \cdot Y - H \int_{K - K'} e^\mu \nu \cdot Y \\ & \quad + \int_S e^\mu (h_\mu - H) \nu \cdot Y \end{aligned}$$

**Remark 3.3.1.** The  $\mu$ -mean curvature  $h_\mu$  arises naturally in the context of principal bundles and seems relevant to more general group actions too. It is for this reason that we carry it throughout our paper.

To elaborate briefly, suppose we have a principal  $G$ -bundle  $G \times P \xrightarrow{\pi} P$ , where  $G$  is a compact Lie group of dimension  $k > 0$ , and  $P$  is a smooth manifold. The orbits of  $G$  then foliate  $P$ , and the quotient  $N := P/G$  is smooth. Moreover, any  $G$ -invariant riemannian metric on  $P$  determines a unique riemannian metric on  $N$  for which the projection  $P \rightarrow N$  is a riemannian submersion.

In this situation,  $N$  acquires a natural **orbit volume function**

$$e^\mu : N \rightarrow (0, \infty), \quad e^\mu(p) := |\pi^{-1}(p)|$$

where  $|\pi^{-1}(p)|$  denotes the  $k$ -dimensional volume of the fiber over  $p$  (cf. [HL]). A routine first-variation calculation then shows:

**Observation 3.4.** *The  $\mu$ -mean curvature  $h_\mu$  of a hypersurface  $\Sigma \subset N$  gives the classical mean curvature  $h$  of its preimage  $\pi^{-1}(\Sigma) \subset P$ .*

One may consequently study  $G$ -invariant hypersurfaces of constant mean curvature  $h \equiv H$  in  $P$  by considering, instead, hypersurfaces of constant  $\mu$ -mean curvature  $h_\mu \equiv H$  in the orbit space  $P/G$ . This can be especially fruitful when  $P/G$  is just two- or three-dimensional.

The requirement that  $G$  act freely on  $P$  seems too restrictive to us. The orbit volume function makes sense for any smooth action of a compact group  $G$  on a riemannian manifold  $P$ . The orbits of highest dimension then foliate an open dense subset  $U \subset P$  (see [MSY]), and the notion of  $\mu$ -mean curvature then extends to  $U/G$  just as it does on the base-space of a principal bundle. Technicalities arise because the full orbit space  $P/G$  typically has (topological) boundary strata of various dimensions, but we think that  $\mu$ -mean curvature could be a useful concept in that setting too.  $\square$

In any event, these facts let us extend the [KKS] conservation law. We continue using the notation above:  $\mathcal{L} \subset \mathcal{I}$  denotes a  $\mu$ -preserving group of isometries on  $N$ , and hence  $L(\mathcal{L})$  comprises the Killing fields that generate its identity component.

**Theorem 3.5** (Conservation law). *Suppose  $\Sigma \subset N$  is an oriented hypersurface with  $h_\mu \equiv H$ , and that  $B$  is a spine for the pair  $(N, \Sigma)$ . Then the formula*

$$(3.8) \quad \phi_B[\mathbf{k}](Y) := \int_\Gamma e^\mu \eta \cdot Y - H \int_K e^\mu \nu \cdot Y$$

*yields a well-defined homomorphism*

$$\phi_B : H_{n-1}(N, \Sigma \cup B) \rightarrow L(\mathcal{L})^* .$$

*Here  $Y$  is any Killing field in  $L(\mathcal{L})$ ,  $K$  is any cap in  $\mathbf{k}$ , and  $\Gamma \in \mathcal{Z}_{n-2}(\Sigma)$  is the unique cycle satisfying  $\partial K + \Gamma \in \mathcal{Z}_{n-2}(B)$ .*

*Proof of Theorem 3.5.* The basic linearity properties of the integral make  $\phi_B$  a homomorphism once we establish well-definition—that

$\phi_B[\mathbf{k}](Y)$  doesn't depend on the  $K \in \mathbf{k}$  we use to compute it. We must show that for all  $Y \in L(\mathcal{L})$ , and all  $K, K' \in \mathbf{k}$ , we have

$$(3.9) \quad \int_{\Gamma} e^{\mu} \eta \cdot Y - H \int_K e^{\mu} \nu \cdot Y = \int_{\Gamma'} e^{\mu} \eta \cdot Y - H \int_{K'} e^{\mu} \nu \cdot Y$$

Here  $\Gamma' \in \mathcal{Z}_{n-2}(\Sigma)$  is the unique cycle for which  $\partial K' + \Gamma' = \partial K + \Gamma$  in  $\mathcal{Z}_{n-2}(B)$ .

We may quickly deduce (3.9) from our constrained first-variation formula (3.7).

Indeed, we assume that  $\mu$  is  $\mathcal{L}$ -invariant,  $Y$  generates a flow that leaves both  $|S|_{\mu}$  and  $|U|_{\mu}$  unchanged, so the left-hand side of (3.7) must vanish. At the same time, the integral over  $S$  on the right of (3.7) also vanishes because we assume  $h_{\mu} \equiv H$ . These facts reduce (3.7) to

$$0 = \int_{\Gamma - \Gamma'} e^{\mu} \eta \cdot Y - H \int_{K - K'} e^{\mu} \nu \cdot Y$$

This is equivalent to (3.9), which is what we needed to show.  $\square$

**Remark 3.5.1.** The simplest case of Theorem 3.5, where  $\mu \equiv 0$  and  $\mathcal{L}$  is the full isometry group of  $N$  (so that  $L(\mathcal{L})$  includes all Killing fields), already generalizes the conservation law in [KKS] by eliminating all homological restrictions on  $N$ .  $\square$

**Remark 3.5.2.** The choice of spine  $B$  in Theorem 3.5 is of no real consequence. For when  $B$  and  $B'$  are both spines for  $(N, \Sigma)$ , the well-definition of  $\phi_B$  on a class in  $H_{n-1}(N, \Sigma \cup B)$  implies that of  $\phi_{B'}$  on a corresponding class in  $H_{n-1}(N, \Sigma \cup B')$ .

To see this, suppose  $\phi_B$  is well-defined on a class  $\mathbf{k}$  containing a cap  $K$  with boundary  $\beta - \Gamma$ , where  $\beta \subset B$  and  $\Gamma \subset \Sigma$ . Then there exists a cycle  $\beta' \subset B'$  homologous to  $\beta$ , and hence an  $(n-1)$  chain  $P$  with

$$\partial P = \beta' - \beta.$$

We claim  $\phi_{B'}$  will now be well-defined on the class  $\mathbf{k}'$  represented by  $K + P$  in  $H_{n-1}(N, \Sigma \cup B')$ .

Indeed, take any cap  $\tilde{K}$  homologous to  $K + P$  in the latter group. Then  $\tilde{K} - P \in \mathbf{k} \in H_{n-1}(N, \Sigma \cup B)$ , and if  $\phi_B$  is well-defined there for some  $Y \in L(\mathcal{I})$ , we have, on the one hand,

$$\phi_B(\tilde{K} - P)(Y) = \phi_B(K)(Y)$$

On the other hand, we have

$$\begin{aligned} \phi_B(\tilde{K} - P)(Y) &= \int_{\Gamma'} e^\mu \eta \cdot Y - H \int_{\tilde{K}-P} e^\mu \nu \cdot Y \\ &= \int_{\Gamma'} e^\mu \eta \cdot Y - H \int_{\tilde{K}} e^\mu \nu \cdot Y + H \int_P e^\mu \nu \cdot Y \\ &= \phi_{B'}(\tilde{K})(Y) + H \int_P e^\mu \nu \cdot Y \end{aligned}$$

Putting these facts together yields

$$\phi_{B'}(\tilde{K})(Y) = \phi_B(K)(Y) - H \int_P e^\mu \nu \cdot Y,$$

Since  $\tilde{K}$  was arbitrary in  $\mathbf{k}'$ , while  $P$  is fixed, we see that  $\phi_{B'}$  is well-defined on  $\mathbf{k}' \in H_{n-1}(N, \Sigma \cup B')$ , as claimed.  $\square$

#### 4. PARTIAL CONVERSE

Suppose the isometry group  $\mathcal{I}$  of our ambient manifold  $N$  contains a closed, connected group  $\mathcal{L}$  preserving a log-density function  $\mu$  as above. Consider an immersed hypersurface  $f : \Sigma \rightarrow N$ , together with a spine  $B$  for the pair  $(N, \Sigma)$ .

Instead of assuming  $\Sigma$  has constant  $\mu$ -mean curvature, we now seek a *converse* to our conservation result, hoping to *deduce* constant  $\mu$ -mean curvature from well-definition of the flux functional  $\phi_B$ . Well-definition of  $\phi_B$ , however, has no meaning without Killing fields on which to test it. Indeed, the potency of assuming well-definition should correlate in some way with the abundance of such fields.

At the same time, we shouldn't have to assume well-definition of  $\phi_B$  on *all* Killing fields. We could restrict  $\phi_B$  to a subset—even just one Killing field—and ask whether well-definition of  $\phi_B$  there influences geometry.

Dually, we shouldn't have to assume constancy  $\phi_B$  on all caps. We have in mind the case where  $\Sigma$  has a continuous (extrinsic) symmetry group  $\mathcal{G} \subset \mathcal{L}$  and  $\phi_B$  takes a fixed value on a sufficiently “crowded” set of homologous  $\mathcal{G}$ -invariant caps.

To make this precise, recall that a chain  $K$  belongs to  $\mathcal{B}_{n-1}(N, \Sigma \cup B)$  if and only if  $K = \partial U - S$  for some  $n$ -chain  $U$  in  $N$  and some  $(n-1)$ -chain  $S$  in  $\Sigma \cup B$ . Since  $B$  has locally finite  $(n-2)$ -dimensional Hausdorff measure, however,  $(n-1)$ -chains in  $B$  are trivial and hence  $\mathcal{B}_{n-1}(N, \Sigma \cup B) \approx \mathcal{B}_{n-1}(N, \Sigma)$ .

**Definition 4.0.3** ( $\mathcal{G}$ -crowded). Call a set of caps  $\mathcal{C} \subset \mathcal{B}_{n-1}(N, \Sigma)$  a  **$\mathcal{G}$ -crowded set of boundaries** if, given any  $\mathcal{G}$ -orbit  $\lambda \subset \Sigma$ , and any  $\varepsilon > 0$ , there exists a cap  $K \in \mathcal{C}$  with  $K = \partial U - S$  for chains  $U$  and  $S$  in  $N$  and  $\Sigma$  respectively, where  $S$  is supported within distance  $\varepsilon$  of  $\lambda$ .

We then say that set  $\mathcal{C}$  of *non*-bounding cycles in  $\mathcal{Z}_{n-1}(N, \Sigma \cup B)$  is a  **$\mathcal{G}$ -crowded set of caps** if the differences  $K - K'$  among  $K, K' \in \mathcal{C}$  form a  $\mathcal{G}$ -crowded set of boundaries.

When  $\mathcal{G}$  is trivial, we simply say **crowded** as opposed to  $\mathcal{G}$ -crowded.

With this definition in hand, we can state and prove a partial converse to our conservation law. Roughly, it says that when flux is conserved across a crowded set of caps — and on Killing fields that commute with  $\mathcal{G}$  — we can split  $\Sigma$  into two nice subsets: One having constant  $\mu$ -mean curvature  $h_\mu \equiv H$ , and one preserved by the commuting Killing fields. The two subsets may overlap, and either can be empty (see Examples 4.5 below).

**Theorem 4.1.** *Let  $\Sigma \subset N$  be an oriented  $\mathcal{G}$ -invariant hypersurface, and  $B$  a spine for the pair  $(N, \Sigma)$ . Suppose  $\mathcal{C} \subset \mathcal{Z}_{n-1}(N, \Sigma \cup B)$  is a  $\mathcal{G}$ -crowded set of caps, and that for each  $Y \in L(\mathcal{L})$ , the  $\mu$ -weighted flux functional*

$$\int_{\Gamma} e^\mu \eta \cdot Y - H \int_K e^\mu \nu \cdot Y$$

*is constant on  $\mathcal{C}$ . Suppose too that  $\mathcal{L}$  centralizes  $\mathcal{G}$ . Then the set*

$$\Sigma_{\mathcal{L}} := \Sigma \setminus h_\mu^{-1}(H)$$

*is  $\mathcal{L}$ -invariant.*

*Proof.* Definition 4.0.3 and the form of the flux functional immediately show that constancy of flux on any  $\mathcal{G}$ -crowded set of caps in  $\mathcal{Z}_{n-1}(N, \Sigma \cup B)$  forces *vanishing* of flux on a  $\mathcal{G}$ -crowded set of *boundaries* in  $\mathcal{B}_{n-1}(N, \Sigma)$ . So without losing generality, we assume  $\mathcal{C}$  to be a crowded set in the latter space.

The heart of our argument then lies with the following

**Claim:** *If  $p \in \Sigma_{\mathcal{L}}$  and  $Y \in L(\mathcal{L})$ , then  $Y_p \in T_p \Sigma$ .*

The definition of  $\Sigma_{\mathcal{L}}$  makes it open in the hypersurface  $\Sigma$ . This ensures that when  $p \in \Sigma_{\mathcal{L}}$ , the latter *separates* any sufficiently small open ball  $B_{\varepsilon}(p) \subset N$  into exactly two connected components. Since  $\mathcal{G}$  is connected and acts isometrically, it preserves  $\Sigma$ ,  $\mu$ , and  $h_{\mu}$ , and hence  $\Sigma_{\mathcal{L}}$ . It follows that  $\Sigma_{\mathcal{L}}$  must also bisect the  $\mathcal{G}$ -invariant tubular  $\varepsilon$ -neighborhood  $B_{\varepsilon}(P)$  of the orbit  $P$  through  $p$ :

$$B_{\varepsilon}(P) := \{gx : x \in B_{\varepsilon}(p), g \in \mathcal{G}\}.$$

$\mathcal{G}$ -crowdedness of  $\mathcal{C}$  now ensures the existence of a trivial cap

$$K_{\varepsilon} := \partial U_{\varepsilon} - S_{\varepsilon}$$

with  $S_{\varepsilon}$  supported in  $\Sigma_{\mathcal{L}} \cap B_{\varepsilon}(P)$ .

Define  $\Gamma_{\varepsilon} := \partial S_{\varepsilon} = -\partial K_{\varepsilon}$ , and substitute  $\Gamma_{\varepsilon}$ ,  $K_{\varepsilon}$  and  $S_{\varepsilon}$  for  $\Gamma$ ,  $K$  and  $S$  respectively in the volume-constrained first-variation formula (3.7). Since  $K_{\varepsilon}$  bounds modulo  $\Sigma$ , we may take  $\Gamma' = K' = \emptyset$  there. For any Killing field  $Y \in L(\mathcal{L})$ , the first two integrals in (3.7) now vanish, because together, they represent the flux across the cap  $K_{\varepsilon} \in \mathcal{C}$ .

The constrained first-variation thus collapses down to one integral:

$$\delta_Y \left( |S_{\varepsilon}|_{\mu} - H |U_{\varepsilon}|_{\mu} \right) = \int_{S_{\varepsilon}} e^{\mu} (h_{\mu} - H) Y \cdot \nu$$

Since  $\mathcal{L}$ , and hence  $Y \in L(\mathcal{L})$  preserve  $\mu$  by assumption, the left side of this identity must *vanish*, so we can rewrite the formula above as

$$(4.1) \quad \int_{S_{\varepsilon}} e^{\mu} (h_{\mu} - H) Y \cdot \nu = 0.$$

From this we will deduce that  $Y \cdot \nu$  vanishes identically on  $S_{\varepsilon}$ , forcing  $Y_p \in T_p S$ , as our Claim requires.

Indeed, suppose toward a contradiction that we had  $Y \cdot \nu \neq 0$  at some  $p \in \Sigma_L$ . In that case, the  $\mathcal{L}$ -invariance and continuity of  $Y \cdot \nu$  ensure that for small enough  $\varepsilon > 0$  we get  $Y \cdot \nu \neq 0$  *throughout*  $\Sigma_{\mathcal{L}} \cap B_{\varepsilon}(P)$ . We assume that  $\mathcal{L}$ —and hence  $L(\mathcal{L})$ —commutes with  $\mathcal{G}$ , so we also get  $Y_{\lambda q} = \lambda_* Y_q$  whenever  $\lambda \in \mathcal{G}$  and  $q \in N$ . But  $\mathcal{G}$  is connected and preserves  $\Sigma$ , so we likewise get  $\nu_{\lambda q} = \lambda_* \nu_q$ . Since  $S_{\varepsilon}$  is connected and remains within  $B_{\varepsilon}(P) \cap \Sigma_{\mathcal{L}}$ , where by definition,  $h_{\mu} - H \neq 0$ , we



conclude that  $(h_\mu - H)Y \cdot \nu$  has a constant non-zero sign on  $S_\varepsilon$ . This contradicts (4.1), though, so it proves our Claim.

To complete the proof of our Theorem, we now combine this Claim with Corollary 2.4.

Recall that  $\mathcal{L}$  acts on  $N$  by isometries, so  $\Sigma_{\mathcal{L}}$  will have locally constant  $\mu$ -mean curvature along its intersection with any orbit  $Q$  of the  $\mathcal{L}$ -action. We want to deduce that  $\Sigma_{\mathcal{L}}$  will contain *all* of each  $\mathcal{L}$ -orbit  $Q$  that it meets. Each orbit  $Q$  is closed, so if

$$\emptyset \neq Q \cap \Sigma_{\mathcal{L}} \neq Q,$$

then  $Q \cap \Sigma_{\mathcal{L}}$  must have a boundary point  $q \in Q$ . At the same time,  $Q \cap \Sigma_{\mathcal{L}}$  is open in  $Q$  by Corollary 2.4, so  $q$  cannot also lie in  $\Sigma_{\mathcal{L}}$ . By completeness of the larger hypersurface  $\Sigma$ , however, we must then have  $q \in \Sigma \setminus \Sigma_{\mathcal{L}}$ , forcing  $h_\mu(q) = H$ . But  $h_\mu$  is locally constant on  $Q \cap \Sigma_{\mathcal{L}}$ , and continuous on  $\Sigma$ . So we must have  $h_\mu = H$  at all points of  $Q \cap \Sigma_{\mathcal{L}}$  sufficiently near  $q \in \partial(Q \cap \Sigma_{\mathcal{L}})$ . This contradicts our definition of  $\Sigma_{\mathcal{L}}$  as the set where  $h_\mu \neq H$ , and thus prohibits  $Q \cap \Sigma_{\mathcal{L}}$  from having any boundary in  $Q \setminus \Sigma_{\mathcal{L}}$ .

It follows that  $Q \cap \Sigma_{\mathcal{L}}$  is both closed, and, by Corollary 2.4, open in  $Q$ . The latter is connected because  $\mathcal{L}$  is, and hence  $Q$  lies entirely within  $\Sigma_{\mathcal{L}}$ . As this holds for any orbit  $Q$  that meets  $\Sigma_{\mathcal{L}}$ , the latter is  $\mathcal{L}$ -invariant, as claimed.  $\square$

**Remark 4.1.1.** We emphasize again that the Theorem above is interesting even when  $\mathcal{G}$  is trivial, in which case it is centralized by any subgroup  $\mathcal{L} \subset \mathcal{I}$ . The Theorem then says, for instance, that when the flux across every sufficiently small trivial cap vanishes on the Killing fields that generate a subgroup  $\mathcal{L} \subset \mathcal{I}$ , the part of  $\Sigma$  that does *not* have constant  $\mu$ -mean curvature  $h_\mu = H$  must be  $\mathcal{L}$ -invariant.  $\square$

**Corollary 4.2.** *If, as in Theorem 4.1, the  $\mu$ -weighted flux functional is constant on a  $\mathcal{G}$ -crowded set of caps, it actually extends as a well-defined conserved quantity to all of  $H_{n-1}(N, \Sigma \cup B)$ .*

*Proof.* While the Theorem assumes constancy of the flux functional  $\phi_B$  only on a  $\mathcal{G}$ -crowded set of caps, the proof then deduces that at every point  $p \in \Sigma$ , either  $h_\mu = H$ , or all Killing fields in  $L(\mathcal{L})$  belong to  $T_p \Sigma$ . In this case, the last integral in the volume constrained first-variation formula (3.3) clearly vanishes on any  $(n-1)$ -chain  $S$  in

$\Sigma$ , so that  $\phi_B(K)(Y) = \phi_B(K')(Y)$  for any two homologous caps  $K, K' \in \mathcal{Z}_{n-1}(N, \Sigma \cup B)$ .  $\square$

**Corollary 4.3.** *If  $N$  is homogeneous,  $\mu$  is constant, and on some crowded set of caps, the flux functional is well-defined for all Killing fields on  $N$ , then  $\Sigma$  has mean curvature  $h \equiv H$ .*

*Proof.* With  $\mathcal{G}$  trivial in Theorem 4.1, well-definition on all of  $L(\mathcal{I})$  lets us take  $\mathcal{L} = \mathcal{I}$ . In a homogeneous space, there are no  $\mathcal{I}$ -invariant hypersurfaces, so  $\Sigma_{\mathcal{L}}$  must be empty, forcing  $h \equiv H$  on  $\Sigma$ .  $\square$

When the  $\mathcal{L}$ -orbits of highest dimension in  $N$  have codimension  $k$ , We say that  $N$  has **cohomogeneity**  $k$  with respect to  $\mathcal{L}$ . A manifold has cohomogeneity zero with respect to some group precisely when it is *homogeneous*.

**Corollary 4.4.** *Suppose  $N$  has cohomogeneity one with respect to a  $\mu$ -preserving group  $\mathcal{L}$ , and on some crowded set of caps, the flux functional is well-defined on all of  $L(\mathcal{L})$ . Then either  $h_{\mu} \equiv H$  on  $\Sigma$ , or else  $\Sigma$  is an orbit of  $\mathcal{L}$ , so that  $h_{\mu}$  is still constant.*

*Proof.* In this situation, the only connected  $\mathcal{L}$ -invariant hypersurfaces are single orbits of  $\mathcal{L}$ , and such orbits clearly have constant  $\mu$ -mean curvature. Since  $\Sigma$  is connected, Theorem 4.1 now yields either  $h_{\mu} \equiv H$ , or that  $\Sigma$  is an orbit of  $\mathcal{L}$ .  $\square$

**4.5. Examples.** Take  $N = \mathbb{R}^3$ , let  $\mathcal{G}$  be the circular group acting by rotation about the  $x$ -axis, and let  $\mathcal{L}$  denote the group that extends  $\mathcal{G}$  by adjoining all translations along the  $x$ -axis. Then  $\mathcal{L}$  commutes with  $\mathcal{G}$  as required by Theorem 4.1. The non-cylindrical Delaunay surfaces—CMC surfaces of revolution about the  $x$ -axis analyzed by C. Delaunay in 1841—show that Theorem 4.1 may obtain with  $\mathcal{G}$ -invariant hypersurfaces having  $h \equiv H$  that *nowhere* inherit the extra symmetries of  $\mathcal{L}$ .

Contrastingly, if we take  $\Sigma$  to be any cylinder centered about the  $x$ -axis with radius *not* equal to  $1/H$ , we get an example with  $\Sigma_{\mathcal{L}} = \Sigma$ . That is,  $\Sigma$  has mean curvature  $H$  nowhere, and yet the flux functional remains well-defined on  $L(\mathcal{L})$ , thanks to the global  $\mathcal{L}$ -invariance of  $\Sigma$ .

Of course, the cylinder of radius  $1/H$  about the  $x$ -axis has *both*  $h \equiv H$  and the extra  $\mathcal{L}$ -symmetry.

All these possibilities show up in the family of twizzlers as well, as we will see shortly.

**4.6. Case study.** (First integrals for twizzlers.) Consider the riemannian product  $N := \mathbb{C} \times \mathbf{S}_R^1$ , where the complex plane  $\mathbb{C}$  and  $\mathbf{S}_R^1$  (the circle of radius  $R$ ) have their standard metrics. Assume a vanishing log-density function  $\mu \equiv 0$ , and take  $\mathcal{G} \approx \mathbf{S}^1$  acting via screw-motion:

$$[e^{it}](z, Re^{i\theta}) = (e^{it}z, Re^{i(t+\theta)})$$

In this situation, each helical orbit of the  $\mathcal{G}$ -action generates  $H_1(N) \approx \mathbb{Z}$ . Let  $\Sigma$  denote a complete connected,  $\mathcal{G}$ -invariant surface in  $N$  (cf. Example 2.5.1), and designate the shortest orbit,  $\beta := \mathbf{0} \times \mathbf{S}_R^1$  as a spine for the pair  $(N, \Sigma)$ .

We can parametrize  $\Sigma$  by applying the  $\mathcal{G}$ -action to an immersed curve  $\gamma : \mathbb{R} \rightarrow \mathbb{C} \times \{1\} \approx \mathbb{C}$  via the map.

$$(4.2) \quad X(u, v) = (e^{iv}\gamma(u), Re^{iv}).$$

Assume that the orientation of  $\gamma$  makes the natural frame  $\{X_u, X_v\}$  *positively* oriented along  $\Sigma$ .

Now fix any point  $p$  on the generating curve  $\gamma$ , and join it to the origin in  $\mathbb{C}$  by a line segment. This segment sweeps out a helicoidal cap  $K^p$ , invariant under the  $\mathcal{G}$ -action, and the homology class of  $K^p$  in  $H_2(N, \Sigma \cup B)$  is clearly independent of  $p$ . One easily sees that as  $p$  varies over  $\gamma$ , the resulting caps  $K^p$  form a  $\mathcal{G}$ -crowded set  $\mathcal{C}$  in the sense of Definition 4.0.3.

Now consider the cylindrical isometry subgroup  $\mathcal{L}$  that extends  $\mathcal{G}$  by including pure  $\mathbb{C}$ -factor rotations of the form  $[e^{is}](z, Re^{i\theta}) = (e^{is}z, Re^{i\theta})$  for any angle  $s$ . This group centralizes  $\mathcal{G}$  and preserves  $\mu$ , as required by Theorem 4.1.

Finally, suppose that for any Killing vectorfield  $Y \in L(\mathcal{L})$ , the flux integral  $\phi_B$  for  $\Sigma$  (relative to the spine  $B = \{\beta\}$  chosen above) is constant on the  $\mathcal{G}$ -crowded set  $\mathcal{C}$  of caps  $K^p$  defined above.

Since  $N$  has cohomogeneity one with respect to  $\mathcal{L}$ , Corollary 4.4 now dictates that *either*  $\Sigma$  is a CMC twizzler with  $h \equiv H$ , or else  $\Sigma$  is an orbit of the  $\mathcal{L}$ -action—a circular cylinder with  $h \equiv 1/r$  (typically  $1/r \neq H$ ,  $r$  giving the radius of the cylinder).

As an application of our theory, we now show that constancy of  $\phi_B$  on the crowded set of helicoidal caps  $K^p$  described above “explains” the first-order ODE known to characterize generating curves of CMC twizzlers, as mentioned in our introduction.

**Proposition 4.7.** *An non-circular immersed curve  $\gamma$  in  $\mathbb{C}$  generates a twizzler in  $\mathbb{C} \times \mathbf{S}_R^1$  with  $h \equiv H$  if and only if it solves*

$$(4.3) \quad \frac{2\pi R^2 (\dot{\gamma} \cdot \mathbf{i} \gamma)}{\sqrt{R^2 |\dot{\gamma}|^2 + (\dot{\gamma} \cdot \gamma)^2}} - \pi R H |\gamma|^2 = c$$

for some  $c \in \mathbb{R}$ .

*Proof.* Since we assume  $\gamma$  is not circular, Corollary 4.4, as noted above, tells us that the flux, across  $K^p$ , of the circular vectorfield  $Y \in L(\mathcal{L})$  given by

$$Y_{(z,\tau)} = -(\mathbf{i}z, 0)$$

will be independent of  $p$  if and only if  $\gamma$  generates a twizzler  $\Sigma$  with  $h \equiv H$ . Equation (4.3) merely evaluates the constancy assertion, namely that

$$(4.4) \quad \phi_B(K^p)(Y) \equiv c \quad \text{for all } p \in \gamma.$$

To get (4.3) from (4.4), we temporarily fix a point  $p = \gamma(t)$  on the generating curve  $\gamma$ , and specify an orientation on the cap  $K^p$ , by declaring the frame field  $\{K_u, K_v\}$  associated with the parametrization

$$K(u, v) = (u e^{iv} p, R e^{iv}), \quad (u, v) \in (0, 1) \times (0, 2\pi),$$

to be positively oriented.

Given this situation, we first consider the second integral in the flux formula (3.8), which pairs  $Y$  with the unit normal  $\nu$  along  $K^p$ . The unit normal to use will be a positive multiple of

$$K_u \wedge K_v = (-R \mathbf{i} e^{iv} p, u |p|^2 \mathbf{i} e^{iv}).$$

The length of  $K_u \wedge K_v$  is actually irrelevant, since we divide by it to normalize, but then multiply it back in as the Jacobian in the flux integral, namely

$$\int_{K^p} \nu \cdot Y = \int_0^{2\pi} \int_0^1 (K_u \wedge K_v) \cdot Y|_{K(u,v)} du dv$$

At  $K(u, v)$ , we have  $Y = -(u i e^{iv} p, 0)$ , so the corresponding flux term evaluates easily to

$$(4.5) \quad H \int_{K^p} \nu \cdot Y = 2\pi H R |p|^2 \int_0^1 u \, du = \pi H R |p|^2$$

With this term in hand, we take up the conormal flux integral, which runs over the curve  $\Gamma$  where  $K^p$  meets  $\Sigma$ . This curve is the helical  $\mathcal{G}$ -orbit of  $p$ , and one easily computes its length as

$$|\Gamma| = 2\pi \sqrt{R^2 + |p|^2}.$$

As  $\Gamma$  forms part of the boundary of an oriented chain  $S$  in  $\Sigma$ , it must *cancel* the corresponding boundary segment of  $K^p$ , since together,  $S$  and  $K^p$  form part of a cycle in  $N$ . A properly oriented parametrization for  $\Gamma$  thus has velocity  $\Gamma'$  equal to a positive multiple of *minus*  $X_v$ . The outer conormal  $\eta$  along  $\Gamma$  must then give the pair  $\{\eta, \Gamma'\}$  positive orientation, so we can obtain  $\eta$  by orthonormalizing *minus*  $X_u$  along  $\Gamma$ , i.e., by normalizing

$$-|X_v|^2 X_u + (X_u \cdot X_v) X_v.$$

Both  $\eta$  and  $Y$  are  $\mathcal{G}$ -invariant, making  $\eta \cdot Y$  constant along  $\Gamma$ , and careful calculation then shows that indeed,

$$\eta \cdot Y \equiv \frac{R^2 \dot{\gamma} \cdot i p}{\sqrt{R^2 + |p|^2} \sqrt{(R^2 + |p|^2) |\dot{\gamma}|^2 - (\gamma \cdot i p)^2}}$$

where we evaluate  $\dot{\gamma}$  at  $p$ . We can simplify the second square root in the denominator here using the elementary identity

$$(\dot{\gamma} \cdot i p)^2 = |\dot{\gamma}|^2 |p|^2 - (\dot{\gamma} \cdot p)^2$$

which allows us to express the first flux integral as

$$(4.6) \quad \int_{\Gamma} \eta \cdot Y = \frac{2\pi R^2 (\dot{\gamma} \cdot i p)}{\sqrt{R^2 |\dot{\gamma}|^2 + (\dot{\gamma} \cdot p)^2}}$$

Setting  $p = \gamma(t)$  and recalling (3.8), we now get  $\phi_B(K^p)(Y)$  by subtracting (4.5) from (4.6). Constancy of flux thus yields (4.3), as predicted.  $\square$

**Remark 4.7.1.** If we parametrize a convex arc of the generating curve  $\gamma$  using its **support function**, namely

$$k(t) := \sup_{\theta} \gamma(t) \cdot e^{i\theta}$$

then

$$\gamma(t) = \left( k(t) + i \dot{k}(t) \right) e^{it}$$

It then follows from Proposition 4.7 that when  $\gamma$  generates a pitch- $R$  twizzler with  $h \equiv H$ , its support function satisfies this simple non-linear ODE:

$$\frac{2Rk}{\sqrt{R^2 + \dot{k}^2}} - H \left( k^2 + \dot{k}^2 \right) = C$$

In other words, the phase portrait of  $k$  lies on one of the “heart-shaped” level curves of the function

$$F(x, y) := \frac{2Rx}{\sqrt{R^2 + y^2}} - H(x^2 + y^2)$$

In [P1] and [P2], Perdomo based his dynamical characterization of twizzler generating curves, and his study of their moduli space, on this observation.

**Remark 4.7.2** (Twizzlers in other 3D-space forms). It is natural to view the curve  $\gamma$  in our case study 4.6 above as the projection of the hypersurface  $\Sigma$  into the *orbit space*  $N/\mathcal{G} \approx \mathbb{C}$ . If we take the orbit-length function in §4.6, given there by  $|\Gamma| = 2\pi\sqrt{R^2 + |p|^2}$ , as a density function  $e^\mu$  in  $\mathbb{C}$ , (cf. Definition 2.1.2), a straightforward reworking of Proposition 4.7 re-interprets the first integral there as the condition for  $\gamma$  to have  $h_\mu \equiv H$  as a “hypersurface” in the two-dimensional orbit space.

Similarly, one can seek CMC “twizzlers” in the 3-sphere  $\mathbf{S}^3 \subset \mathbb{R}^4$  that are invariant under one of the helical  $(k, l)$  “torus knot”  $\mathcal{G} = \mathbf{S}^1$  actions on  $\mathbf{S}^3$  given by

$$[e^{it}](z, w) = (e^{ikt}z, e^{ilt}w)$$

This is the standard Hopf action when  $k = l = 1$ , in which case the orbit space  $\mathbf{S}^3/\mathcal{G}$  is of course the standard 2-sphere  $\mathbf{S}^2$ . More generally, when  $\gcd(k, l) = 1$ , one can realize the orbit space as an eccentric “football” shaped surface of revolution in  $\mathbb{R}^3$ , smooth except for conical singularities at one or both ends. The  $\mathcal{G}$ -invariant CMC twizzlers in  $\mathbf{S}^3$  then correspond one-to-one with curves having constant  $\mu$ -mean curvature in the orbit space, where  $\mu(p) = \ln |\pi^{-1}(p)|$  for each  $p$  in the orbit space. By Theorem 4.1, these are precisely the non-circular curves that conserve flux along the Killing fields that generate the rotational symmetry of the orbit space. It is then straightforward

to use this fact, as in Proposition 4.7, to derive the first integral for such curves. See [E] for the resulting expression.

Analogous helical actions exist in the hyperbolic space form  $\mathbb{H}^3$ , and the resulting CMC twizzlers have a first integral that one can derive in precisely the same way. The reader may consult [E] for a description of the group action and the resulting first integral in this case as well.

### ACKNOWLEDGMENTS

The Summer 2011 Math REU program at Indiana University, Bloomington initiated and supported this work. We gratefully acknowledge the National Science Foundation for funding that program.

### REFERENCES

- [DD] DoCarmo & Dajczer, *On helicoidal surfaces of constant mean curvature*, Tohoku Math. J. **2** Volume 34, Number 3 (1982), 425-435.
- [E] Edelen, N., *A conservation approach to helicoidal constant mean curvature surfaces in  $\mathbb{R}^3$ ,  $\mathbb{S}^3$  and  $\mathbb{H}^3$* , arXiv preprint 1110.1068
- [GMT] Federer, H. **Geometric Measure Theory**, Springer-Verlag, New York 1969.
- [H] Halldorsson, *Helicoidal surfaces rotating/translating under the mean curvature flow*, arXiv preprint 1106.4583
- [HL] Wu-Yi Hsiang & H.B. Lawson, *Minimal submanifolds of low cohomogeneity*, J. Diff. Geom. **5** (1971) 1–38
- [KKS] N. Korevaar, R. Kusner & B. Solomon, *The structure of complete embedded surfaces of constant mean curvature*, J. Diff. Geom. **30** (1989), 465–503
- [MSY] D. Montgomery, H. Samelson, & C.T. Yang, *Exceptional orbits of highest dimension*, Ann. of Math., **64** (1956) 131–141
- [SL] L. Simon, **Lectures on Geometric Measure Theory**, Proceedings of the Centre for Mathematical Analysis, Vol. 3, 1983, Australian National University, Canberra
- [SM] A. Stehney & R.A. Millman, *Riemannian manifolds with many Killing vector fields*, Fund. Math. **105** (1979/80), no. 3, 241–247
- [P1] O. Perdomo, *A dynamical interpretation of the profile curve of cmc twizzlers*, arXiv preprint 1001.5198
- [P2] O. Perdomo, *The treadmill of a curve*, arXiv preprint 1105.3460

- [W] W. Wunderlich, *Beitrag zur Kenntnis der Minimalschraubflächen*, Compositio Math, **10** (1952), 297–311.

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